WHAT OUGHT THE STUDY OF MATHEMATICS TO CONTRIBUTE TO THE EDUCATION OF THE HIGH-SCHOOL PUPIL?[1]

PUPILS in a high school seem naturally to divide into two classes, those who end their formal education there, and those who are planning definitely for a college course. It is true that a large percentage of our high-school pupils never attend school after leaving us, while a few continue definite study for some time longer. Whether for each of these a different course of study should be pursued — a different training offered, is still an open question not in the province of this paper to discuss. But whatever difference of opinion may exist as to the studies for each class to pursue — whether they should be for one educational and for the other so-called practical, whenever mathematics is taken up at all, there is quite general agreement that the culture aim should be the same. In this paper we shall, then, treat the high school as a unit.

Arithmetic, algebra, and geometry, with trigonometry for a few in a few schools, constitute the formal high-school work in mathematics.

Were my subject, "The results which we do not reach in the teaching of arithmetic," I could read you a long paper and one eloquent with the deepest feeling. Each September finds me ready to begin the review of arithmetic with my one hundred forty and more seniors. The days are full of hope, and faith is strong that *this time* the outcome will be all that I desire. Each succeeding February brings the written proof of unsatisfactory results, and I am consumed with remorse, only to arise in the fall to renewed hope and its sure-following disappointment.

Perhaps the results that should be reached in arithmetic may be inferred from an enumeration of my own failures.

[1] Proceedings of the Michigan Schoolmasters' Club.

My pupils are not accurate in computation, they do not hold themselves responsible for such accuracy; have no habit of persistently checking their work at each step to be sure of themselves by the way. They are afraid of a fraction. If the result fails "to come out even," they are in despair. If they chance to discover that others in the class get the same result, they take courage, but their own work does not carry conviction, for they have not learned to rely upon their own judgment. They are not sensitive to the reading of a problem and consequently have no basis for a logical method of attack. They fail to see that arithmetic is a part of a great mathematical whole, or even that arithmetic itself is a unified subject based upon a definite and consistent system of notation. They rather look upon it as an aggregation of isolated subjects for the mastery of each of which a separate trick must be learned and remembered. If memory fails, they have no resource but to juggle with the example according to the various tricks which they do remember until they hit upon the right combination.

These are my failures, frankly admitted, but the cause of them is farther to seek than my own faulty teaching three times a week for twenty weeks.

The Grand Rapids schools are doubtless representative in their work below the high school. The study of arithmetic is begun in the fourth grade, continued throughout the grades, prolonged through the first half of the ninth year, and culminates in a three-fifths course for the seniors. This gives to the pupils five and a half years consecutive study of the same text-book, with the promise of a farewell interview later on. In view of this state of things, should we not have less arithmetic in the lower grades, or, at least, less arithmetic that is beyond the logical grasp of the pupils at the time it was given?

The pupils have had arithmetic poured into them and over them; they have been submerged in arithmetic until, when they reach the high school, they are sodden with it. But even then the first act of the high school is to plunge them again into arithmetic, thus destroying the last chance of getting them

dried out into good arithmetical timber by the time of graduation.

If the solution does not lie in a diminution of quantity, then it must come through more efficient teaching in the grades. Just where the fault is must be determined by the supervisors of that work, but something is wrong somewhere, and until the evil is remedied, our high schools can never give to their pupils the education in this branch that is their due. About all that can be accomplished is to arouse them to some sense of their deficiencies, point out the right way, and trust to bitter experience later on to correct the fault.

As to our own school, I am pleased to say that our superintendent is at present conducting some interesting experiments in the grades, looking toward the desired end, so with us there is hope of better things in the near future.

Leaving arithmetic, we enter upon a field of mathematics in which the high school has full proprietorship and consequent responsibility.

Later years have marked a change in the attitude of educators toward the subject-matter to be taught in algebra and geometry. The colleges are clamoring for wider technical knowledge ; they are asking for a change from "the traditional algebra of most of our secondary schools to such instruction as shall make the transition to the reconstructed algebra of the best American colleges less abrupt." In geometry strong pressure is brought to lead into new fields, and we are asked either to accept modern geometry pure and simple or to modernize our old geometry. That this is true is evinced by the number of books that have recently been put upon the market, the prefaces to which all indicate a demand upon the part of mathematicians for such work.

We should listen to all this and give it careful, but conservative, attention. Every teacher of mathematics in our secondary schools will eventually find it necessary to settle this question. Such lines of work should be given to the pupil as shall offer the best all-around discipline, but there is no reason why this same discipline should not accompany strong mathematical work.

The high school has vital reasons for existence other than its preparatory relations to colleges, and its purpose is not primarily to make mathematicians any more than to make specialists in other directions yet, as one of the recent bookmakers has said: "Quite as good exercise may be found in getting somewhither as in walking around the square."

Whatever the subject-matter selected, there should come from the study of mathematics accuracy of work, facility in handling mathematical quantities, and the best possible technical training as far as one goes. If the best results are reached mathematically, there will go with them a certain ethical training which cannot be separated from good work. Good work cannot exist without carrying with it, though unconsciously, perhaps, this same ethical training. No other line of study in our curriculum offers such valuable discipline as does this seemingly dry subject. The classics may approach it and in certain lines be equally valuable, but for a complete setting in order of one's intellectual house, mathematics has no rival. Arithmetic, algebra, and geometry all have their share in the good work, but by none of them can the claim be so well substantiated as by geometry.

It is not in the province of a high school to teach a trade, to produce expert business men, to fit for a profession, or, in fact, to specialize in any direction. Our duty to our boys and girls is to give them such training as shall make possible their development into the best and broadest men and women consistent with the conditions that may surround them after they leave us. If we give to a pupil the right *training*, whether he goes to business or to college, the mastery that he has gained over his powers will enable him to take up the life of either place creditably. The technicalities of business he will soon master, and the wisdom of the school men will appeal to an eager student.

Let us examine for a little the results that mathematics should attain in this direction.

"Order is heaven's first law," and the mathematician's as well, but young people do not naturally fall in with its requirements. Their enthusiasm leads them on with reckless haste to the one

purpose before them, and things are thrown hither and thither to clear the way at the moment. Mathematics is a formalist in the best sense of the word, and when the subject is rightly taught, pupils will unconsciously fall into general orderly habits. They will see, too, that form is not all a hollow show, but that it has a clearly defined value. Pupils who have the habit of solving examples in a slovenly, confused manner rarely stand high in their class. Of course, there will be found here and there a genius, and geniuses know no law, but, in the main, good form in the solution of an example means clear mathematical insight, and such form is often an actual help to correct solution. Different types of examples have each a different physiognomy, and when this is distorted by the caricature of deformed work, the relation to the type is lost. The right study of equations should lead to a ready recognition of such typical relations, and at consequent ready handling of them. Upon this foundation rests the whole matter of formulæ so important in higher mathematics. If form is the foundation of formulæ, it must not be undervalued at the begining of a mathematical course.

In geometry the same care for form should be continued. The time is no more when an untidy, carelessly drawn figure can pass muster with a good teacher, and the argument that painstaking drawing absorbs too much time, is no longer valid. At the beginning, the forming of correct habits may be slow, but, those once established, a pupil will draw a good figure in as little time as an untutored one will take to perpetrate a careless one. A class that is trained to good drawing cannot help advancing more rapidly, for a carefully executed figure will often reveal a mathematical truth which an ill-constructed one only conceals. When a figure is right and clear, it readily appeals to the eye, and the eye, becoming thus the servant of the brain, assists in reaching conclusions, while with a poor figure the brain must struggle to put upon the distorted image the interpretation that shall harmonize with the mental concept.

The orderly workman is the rapid workman, and it is the right of young people to be trained to orderly habits. The

school responsibility in this direction rests largely upon the teachers of mathematics, and I believe that any school in which careful work in algebra and geometry is insisted upon will show, in all directions, the effect of such influence.

That I do not exaggerate the importance of this side of the subject, is witnessed by the new texts in geometry, the best of which call particular attention to the perfection and attendant clearness of the figures in the book, and urge that the pupil shall imitate them.

Again, who has not repeatedly received to a question the answer: "I know but I cannot tell"? The regulation response is: "If you know, you can tell what you know." This is not the truth, for may not one have a reasonably clear mental concept and yet be unable to translate into good English? Children talk in suggestions, using merely a word to label the idea, trusting to your knowledge of language to supply the rest. The high-school pupil will do the same if allowed to, and unless the teacher is ever alert, a class may complete the work and know some mathematics, too, without ever being trained to sharp expression. The accurate definitions of geometry offer the best possible opportunity for drill in this direction, and no training in high-school mathematics is adequate which does not give to the pupil this education in clear expression. The habit thus formed will follow through life, and will give an honesty of expression that the American people stand greatly in need of.

For a long period the mind is receptive only and gives back what it receives in much the same form in which it was imparted to it. The young mind is unthinking and speaks from impulse, not from conviction. But if the development is rightly conducted, the pupil must be roused to watch his own mind, to discover what is actually going on inside his own head, to have faith in his own mental convictions, and the fearlessness to express them.

Do you remember the dialogue that Plato represents as taking place between Socrates and Meno? To illustrate a certain point in their discussion, Socrates has one of Meno's attendants

summoned before him. The boy was a Greek and presumably the average serving lad of the time.

Socrates asks him various questions about the square. The first few he answers correctly and with assurance. Encouraged by his own brilliancy, he innocently follows Socrates' lead. The conclusion to which he finally brings the boy is that a double square comes from a double line. So sure is the boy of his conclusion that he gives now his first answer of any length:

"Certainly, Socrates, that will be double."

Turning to Meno, Socrates says: "And now he fancies that he knows how long a line is necessary to produce a figure of eight square feet. . . . . And does he really know? . . . . He only guesses that because the square is double the line is double."

Then Socrates patiently retraces the steps of the argument. The first sign of enlightenment that the boy reveals is when he meditatively replies to one of Socrates' questions:

"Yes, that is what I think."

"Very good," Socrates encouragingly replies. "I like to hear you say what you think."

Then the instruction continues until the boy begins to reply in the negative, and at last gives the answer towards which the great teacher has been driving him:

"Indeed, Socrates, I do not know."

Again addressing Meno, Socrates says, "Do you see what advances he has made in the power of recollection? He did not know at first, and he does not know now . . . . but he thought he knew, and had no difficulty, but now he has a difficulty, and neither knows nor fancies that he knows. . . . Is he not better off in knowing his ignorance? . . . . If we have made him doubt and given him the 'torpedo's shock,' have we done him any harm? . . . . Now he will wish to remedy his ignorance, but then he would have been ready to tell all the world that the double space should have the double side. Then he was the better for the torpedo's touch."

Geometry is the "torpedo" that should swim vigorously in the high-school sea. When you have made a pupil think of all you

can count it great gain, for even the thinking that may arrive at wrong conclusions is better for the mind than stagnation. You have made progressive strides when you see your pupil thinking independently, and talking out to you his conclusions fearlessly. The best teacher of geometry that I know says that the right results have not been attained until a pupil not only ceases to lean upon the teacher, but is able to stand up against a teacher and argue his case with confidence and in a manner to carry conviction to his hearers.

Of all the high-school studies, geometry alone offers drill in formal logic, so it is here that the mind must be trained to a knowledge of its powers and the proper use of them. Every demonstration should be a faultless argument. The pupil should be taught to ground himself thoroughly upon the hypothesis and then work toward the conclusion by honest logical steps. Through the influence of correct demonstration he will learn to feel the relation of cause and effect, of act and consequence, and to realize the limitations of human power.

Let us trace in a general way the steps of such a demonstration. In the first place, the hypothesis must be carefully isolated. He must see to it that no extraneous matter creeps into it, and that he claims as his own only what honestly belongs to him by gift of the proposition stated, and he must see to it that his argument is based upon the hypothesis. Next he must examine his resources, take an account of his stock in trade, and determine what he can bring of previously tested knowledge to aid in the proof which he has undertaken. Probably he will first say: "Draw such and such lines." Then will come the inevitable "Why?" asked first by the book, repeated many times by the teacher, until the pupil learns to stand sentinel over his own mind, and, challenging every step of his progress, is "ready to give an answer to every man that asketh a reason."

After the figure is constructed, the pupil must examine it. He may discover in it a dozen things, every one of which is true, but only two or three of them have any concern with the business in hand. Let his mind be so trained that these will seem to

spring toward him and all the rest fade into the dimness of indifference. Such training will teach him to select from any group of facts or conditions the central one. It will teach him to seize upon the vital feature in any relation of life. Such a mind will take the tide at its flood and rarely find its life bound in shallows and disasters.

Perhaps in the demonstration it becomes desirable to apply one part of the figure to another. He may lift that part and place one line of it or one angle upon its equal, but there his control over the matter ceases. From that moment he is under law and face to face with the eternal. Let him stand there a humble spectator, knowing that, "Till heaven and earth pass, one jot or tittle shall in no wise pass from the law" over which his finite mind has no control. He was responsible for placing the figures together, but before the resulting consequences he is helpless. He may watch and see if in the finality there is aught that concerns him. Here again he will discover various things accomplished, but again he must recognize the one conclusion for which he has been striving, must isolate it from the rest and hold it up to view with the strength of conviction.

Thus far we have discussed the cold, intellectual side of mathematics, but when that is done the all is not said. You are keeping the minds of your pupils in a prison house if you do not help them burst the bonds of sense and rise toward the infinite.

Mathematics may toil in the mines, it may float upon the ocean the ships of war and of commerce, it may perform any service that man demands. "For whoever will be chief among you, let him be servant of you all." This service, however, does not bind to earth nor to things visible. We begin with the concrete, but unless the imagination is so cultivated that it can eventually sweep out far away from all this, and trace the relation of the unseen to the visible, geometry has failed of its full mission. The things of time and sense are verities, but beyond lie the beauteous palaces of the imagination built upon a sure foundation and rising into the limitless expanse of space. Help your pupils to build these palaces, and into them will silently

glide the spirit of poetry to dwell there forever. Geometry seems to stand for all that is practical, poetry for all that is visionary, but in the kingdom of the imagination you will find them close akin, and they should go together as a precious heritage to every youth.

The picture that Wordsworth gives us in "The Prelude" is always charming to me. Musing one day by the seashore

> On poetry and geometric truth
> And their high privilege of lasting life,

he fell asleep and dreamed a dream. Over a vast wilderness an Arab passed,

> A lance he bore, and underneath one arm
> A stone, and in the opposite hand a shell
> Of surpassing brightness.

He was hasting inland to bury these treasures from the pursuing waters of the deep. The precious stone was geometry and the shell its sister treasure, poetry — the two things in the world that must be saved:

> The one that held acquaintance with the stars,
> And wedded soul to soul in purest bond
> Of reason undisturbed by space or time.
> The other that was a god.

Geometry, in the hands of an inspired teacher, is not alone of this world, but it sweeps to the infinite realm occupied only by the pure imaginings of the geometer and the clear visionings of the poet.

To sum up, then, the study of mathematics should add three things to the education of the high-school pupil:

It should give him thorough training in the facts of mathematics so far as the subject is taken up, and should give facility in the handling of its various functions in computations.

From the ethical side, it should teach him order, give him integrity of thought, honesty of expression, ability to isolate the central point in any issue, and quickness, under all circumstances, to seize upon the thing of vital importance to him at the moment.

From the poetic side, he must learn to be at home in

> "An independent world
> Created out of pure intelligence."

A boy or girl trained in this wise will be no weakling. He will know his strength, he will recognize his limitations, and he will have a reverent respect for the eternal verities of the universe.

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